Practical Applications

The computer program has been of considerable assistance in correcting problems of new rocket-engine hardware. It has been used to diagnose correctly and eliminate start problems in both ignited and hypergolic start sequences using both cryogenic and storable propellants.

Although there have been no experiments available for comparison, no trouble has been encountered in calculating cycles of pulse-mode operation (Fig. 7).

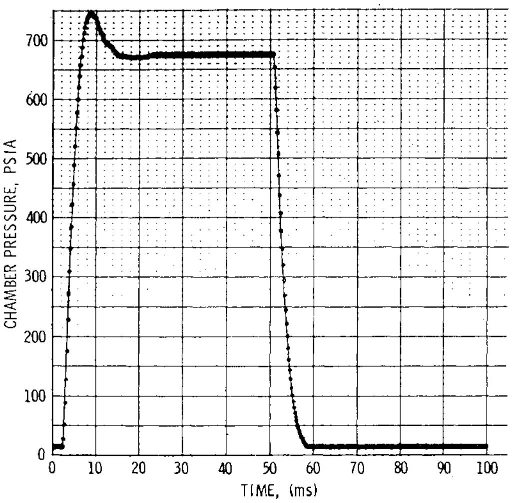

Fig. 7 Pulse-mode chamber pressure vs time.

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Stability of Motion of Force-Free Spinning Satellites with Flexible Appendages

LEONARD MEIROVITCH*
Virginia Polytechnic Institute and State University, Blacksburg, Va.
AND
ROBERT A. CALICO†
University of Cincinnati, Cincinnati, Ohio

This paper presents two approaches to the stability analysis of torque-free spinning bodies consisting of a main rigid body and a number of distributed elastic parts. The stability analysis is based on the Liapunov direct method and takes into consideration automatically the existence of motion integrals. The first approach to the stability problem is based on modal analysis, whereas the second one makes use of integral coordinates. The case of a torque-free satellite represented by a rigid hub with six flexible appendages is solved. Closed-form stability criteria derived by the second approach compare favorably with numerical results obtained by modal analysis.

Introduction

THE rotational motion of a torque-free rigid body is known to be stable if the rotation takes place about an axis corresponding to either the maximum or the minimum moment of inertia, but the motion is unstable if the rotation takes place about the axis of intermediate principal moment of inertia [for example, Meirovitch (Ref. 1, Sec. 6.7)]. In general, however, spacecraft are not entirely rigid and the question remains as to what extent the rigid-body idealization can be justified.

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* Professor, Department of Engineering Mechanics. Formerly Professor, Department of Aerospace Engineering, University of Cincinnati. Member AIAA.

† Instructor, Department of Aerospace Engineering. Member AIAA.

Addressing themselves to the flexibility effects, Meirovitch and Nelson[2] investigated the stability of motion of a satellite containing elastic parts by means of an infinitesimal analysis. Reference 2 treats distributed elastic members by means of both the spatial discretization method and modal analysis, where in the latter the effect on the system stability of truncating the series is explored. Also related to the present problem is the problem of a satellite with elastically connected moving parts investigated by Nelson and Meirovitch[3] via the Liapunov direct method. The dynamics of satellites containing elastic parts was also studied by Likins and Wirsching[4] by means of a discrete model.

The Liapunov direct method has been widely used to analyze the stability of discrete systems. In recent years, however, work has been done on extending the Liapunov method to distributed-parameter systems. In this regard, we single out the works by Wang[5,6] and by Parks[7] who applied the method to analyze the stability of partial differential equations associated with elastic and aeroelastic systems. In Ref. 5 Wang presents a simple example of a hybrid system, namely a system described by sets of both ordinary and partial differential equations. In a first attempt to apply Liapunov's direct method to hybrid systems from the area of satellite dynamics, Meirovitch[8,9] studied the stability of a spinning rigid body with elastic appendages. Several new concepts were introduced in Ref. 8, such as the use of the bounding properties of Rayleigh's quotient to eliminate the dependence of the testing functional on the spatial derivatives, as well as the concept of a testing density function. Reference 9 extends the theory to torque-free hybrid systems.

The present study extends the work of Refs. 8 and 9 to the case of hybrid systems in which testing density functions cannot be readily defined. The mathematical model consists of a torque-free spinning rigid body with three pairs of rigidly attached flexible rods. The initial steps of the stability analysis follow the pattern of Ref. 8, in which it is shown that under certain circumstances the system Hamiltonian H is a suitable Liapunov functional. In contrast to the method of Ref. 8, however, difficulty is encountered in defining an appropriate testing density function. Two approaches are presented here to circumvent this difficulty. The first, modal analysis in conjunction with series truncation, leads to stability criteria in terms of finite series. The second method involves defining new time-dependent coordinates in terms of certain integrals appearing in the system Hamiltonian. Using these "integral coordinates" and Schwarz's inequality for functions it is possible to discretize the testing functional and test its sign properties by using Sylvester's criterion. This method yields closed-form stability criteria lending themselves to ready physical interpretation.

General Problem Formulation

The problem formulation is essentially that of Ref. 9. We shall present here the main features only, as the interested reader can find the details in Ref. 9.

Let us consider a body of total mass m moving relative to an inertial space XYZ (Fig. 1 of Ref. 9), and define two sets of body axes, the set xyz with the origin at point 0 and coinciding with the principal axes of the body in the undeformed state, and the set $\xi\eta\zeta$ which is parallel to xyz but has the origin at the center of mass c of the deformed body. The position of a typical point in the undeformed body relative to axes xyz is denoted by the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the elastic displacement of an element of mass dm, originally coincident with that point, by the vector $\mathbf{u} = u(x,y,z,t)\mathbf{i} + v(x,y,z,t)\mathbf{j} + w(x,y,z,t)\mathbf{k}$, where $\mathbf{i,j,k}$ are unit vectors along axes x,y,z (or axes ξ,η,ζ), respectively. The radius vector from point 0 to c is given by $\mathbf{r}_c$, so that $\mathbf{u}_c = \mathbf{u} - \mathbf{r}_c = u_c\mathbf{i} + v_c\mathbf{j} + w_c\mathbf{k}$ represents the displacement vector measured with respect to axes $\xi\eta\zeta$. Assuming that axes xyz, hence also axes $\xi\eta\zeta$, rotate with angular velocity $\boldsymbol{\omega} = \omega_\xi\mathbf{i} + \omega_\eta\mathbf{j} + \omega_\zeta\mathbf{k}$ relative to the inertial space, and denoting by $\dot{\mathbf{u}}_c' = \dot{u}_c\mathbf{i} + \dot{v}_c\mathbf{j} + \dot{w}_c\mathbf{k}$ the velocity of dm relative to $\xi\eta\zeta$ due to the elastic effect, we have $\dot{\mathbf{r}} + \dot{\mathbf{u}}_c = \dot{\mathbf{u}}_c' + \boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}_c)$.

Omitting the details, the kinetic energy can be written conveniently in terms of matrix notation

$$T = \tfrac{1}{2}m\{\dot{R}_c\}^T\{\dot{R}_c\} + \tfrac{1}{2}\{\omega\}^T[J]\{\omega\} + \{K\}^T\{\omega\} + \tfrac{1}{2}\int_m \{\dot{u}_c'\}^T\{\dot{u}_c'\}dm \tag{1}$$

where $\{\dot{R}_c\}$ is the column matrix whose elements are the velocity components of c, $\{\omega\}$ the column matrix corresponding to $\boldsymbol{\omega}$, $[J]$ the inertia matrix of the body in deformed state, $\{K\}$ a column matrix representing the angular momentum due to elastic velocities, and $\{\dot{u}_c'\}$ the column matrix of these elastic velocities. The elements of $[J]$ and $\{K\}$ are given in Ref. 9.

The gravitational potential energy is assumed to be very small compared with the kinetic energy, or the elastic potential energy, and will be ignored. The elastic potential energy, denoted by V_{EL}, depends on the nature of the elastic members and can be regarded as depending on the partial derivatives of u_c,v_c,w_c with respect to x,y,z. In particular, it is assumed that V_{EL} is a function of spatial derivatives through second order.

An application of Hamilton's principle leads to the system Lagrangian equations of motion, which can be converted into twice the number of Hamiltonian equations. Details of the derivation are given in Ref. 8 and will not be repeated here, although it should be mentioned that the first term on the right side of Eq. (1) was ignored under the assumption that it is constant. Quoting directly from Ref. 8, we write the Hamiltonian equations of motion

$$\dot{\theta}_i = \partial H/\partial p_{\theta_i},\ \dot{p}_{\theta_i} = -\partial H/\partial \theta_i,\ i = 1,2,3 \tag{2a}$$

$$\left.\begin{aligned}
&\dot{u}_c = \partial\hat{H}/\partial\hat{p}_{u_c},\ \dot{v}_c = \partial\hat{H}/\partial\hat{p}_{v_c},\ \dot{w}_c = \partial\hat{H}/\partial\hat{p}_{w_c}\\
&\dot{\hat{p}}_{u_c} = -(\partial\hat{H}/\partial u_c) + \mathscr{L}_{u_c}[u_c,v_c,w_c] + \hat{Q}_{u_c}\\
&\dot{\hat{p}}_{v_c} = -(\partial\hat{H}/\partial v_c) + \mathscr{L}_{v_c}[u_c,v_c,w_c] + \hat{Q}_{v_c}\\
&\dot{\hat{p}}_{w_c} = -(\partial\hat{H}/\partial w_c) + \mathscr{L}_{w_c}[u_c,v_c,w_c] + \hat{Q}_{w_c}
\end{aligned}\right\}\text{at every point of } D_e \tag{2b}$$

where D_e represents the domain of the elastic continuum. Equations (2b) are subject to the boundary conditions

$$\mathbf{B}_j[u_c,v_c,w_c]\cdot\mathbf{B}_k[u_c,v_c,w_c] = 0 \text{ on } S(j = 1,2;\ k = 3,4) \tag{3}$$

in which S is the surface bounding D_e. Only two of the combinations in Eq. (3) must be satisfied at every point of S, where the indices j and k are different for each combination. The quantities $\mathscr{L}_{u_c},\mathscr{L}_{v_c},\mathscr{L}_{w_c},\mathbf{B}_j$, and $\mathbf{B}_k$ are differential operators defined in Ref. 8. Moreover

$$p_{\theta_i} = \partial L/\partial\dot{\theta}_i,\ i = 1,2,3,\ \hat{p}_{u_c} = \partial\hat{L}/\partial\dot{u}_c,\hat{p}_{v_c} = \partial\hat{L}/\partial\dot{v}_c,\ \hat{p}_{w_c} = \partial\hat{L}/\partial\dot{w}_c \tag{4}$$

are generalized momenta, where the last three are momentum densities. The functional L in Eq. (4) is the Lagrangian which has the general form

$$L = T - V_{EL} = \int_D \hat{L}(\theta_i,\dot{\theta}_i,u_c,v_c,\ldots,\dot{w}_c,\partial u_c/\partial x,\partial u_c/\partial y,\ldots,\partial w_c/\partial z,\partial^2 u_c/\partial x^2,\partial^2 u_c/\partial x\partial y,\ldots,\partial^2 w_c/\partial z^2)dD \tag{5}$$

in which $\hat{L}$ is the Lagrangian density. In addition, the quantities $\hat{Q}_{u_c},\hat{Q}_{v_c},\hat{Q}_{w_c}$ represent distributed internal damping forces which depend on the elastic motion alone and not on the rotational motion. The Hamiltonian H is defined by

$$H = \int_{D_e}\hat{H}dD_e = \sum_{i=1}^{3}p_{\theta_i}\dot{\theta}_i + \int_{D_e}(\hat{p}_{u_c}\dot{u}_c + \hat{p}_{v_c}\dot{v}_c + \hat{p}_{w_c}\dot{w}_c)dD_e - L \tag{6}$$

and $\hat{H}$ is the corresponding Hamiltonian density. It should be noticed here that the Hamiltonian has a hybrid form as it is a function and a functional at the same time.

Stability Analysis of Hybrid Dynamical Systems

A general and rigorous method for the stability analysis of hybrid systems of equations has been developed in Ref. 10. We shall not present all the details here but only summarize the main features.

Consider a hybrid system with the state vector given by $\mathbf{v} = \mathbf{v}_d(t) + \mathbf{v}_c(P,t)$, where $\mathbf{v}_d(t)$ and $\mathbf{v}_c(P,t)$ represent discrete and continuous variables, respectively. The system is described by the set of differential equations

$$\dot{\mathbf{v}} = \mathbf{V}(\mathbf{v}, \partial\mathbf{v}_c/\partial x, \partial\mathbf{v}_c/\partial y, \ldots, \partial\mathbf{v}_c{}^{2p}/\partial z^{2p}) \tag{7}$$

where $\mathbf{V}$ is a vector function depending on the state vector and spatial derivatives of the state vector through order $2p$, in which p is an integer. The continuous variables must also satisfy appropriate boundary conditions. The state vector can be imagined geometrically as representing an element in a space S which can be regarded as the cartesian product of a finite dimensional vector space and a function space, the first corresponding to $\mathbf{v}_d$ and the second associated with $\mathbf{v}_c$.

To test the stability of system (7) in the neighborhood of the trivial solution, we define a scalar functional $U = U(\mathbf{v}, \partial\mathbf{v}_c/\partial x, \partial\mathbf{v}_c/\partial y, \ldots, \partial^p\mathbf{v}_c/\partial z^p)$ such that $U(\mathbf{0},\mathbf{0},\ldots,\mathbf{0}) = 0$. We note that U depends on spatial derivatives through order p, as opposed to $\mathbf{V}$ which depends on derivatives through order $2p$. Moreover, the total time derivative of U along a trajectory of the system is defined by

$$\dot{U} = dU/dt = \nabla U_d \cdot \dot{\mathbf{v}}_d + \int_D \nabla \hat{U}_c \cdot \dot{\mathbf{v}}_c dD$$
$$= \nabla U_d \cdot \mathbf{V}_d + \int_D \nabla \hat{U}_c \cdot \mathbf{V}_c dD$$

where the subscripts d and c designate quantities pertaining to discrete and continuous variables, respectively.

In Ref. 8 four theorems, representing extensions to hybrid systems of Liapunov's and Krasovskii's theorems, are presented. They all require the knowledge of the sign properties of U. Since the scalar functional U depends on spatial derivatives of $\mathbf{v}$, it may be difficult at times to determine its sign properties. In such cases it may be possible to define another scalar functional $W(\mathbf{v})$, depending on the state vector $\mathbf{v}$ alone, and such that $U \geq W$. Then we can state the following:

Stability Theorem

Suppose that for system (7) there exists a scalar functional U such that $\dot{U}$ is negative semidefinite along every trajectory of Eq. (7) and, in addition, the set of points at which $\dot{U}$ is zero contains no nontrivial positive half-trajectory. Then, if a positive definite functional W can be found such that $U \geq W$, the trivial solution $\mathbf{v} = \mathbf{0}$ is asymptotically stable.

Taking the total time derivative of H it is shown in Ref. 8 that

$$\dot{H} = \int_{D_e} (\hat{Q}_{uc}\dot{u}_c + \hat{Q}_{vc}\,\dot{v}_c + \hat{Q}_{wc}\dot{w}_c) dD_e \tag{8}$$

Next we assume that the damping forces are such that $\dot{H}$ is negative semidefinite, $\dot{H} \leq 0$. Moreover, due to coupling, the forces $\hat{Q}_{uc}, \hat{Q}_{vc}, \hat{Q}_{wc}$ are never identically zero at every point of the phase space but they reduce to zero at an equilibrium point. Hence, if the Hamiltonian H is positive definite at an equilibrium point, then, by Theorem 2 of Ref. 8, H can be regarded as a Liapunov functional and the equilibrium point under consideration is asymptotically stable. On the other hand, if H is not positive definite and there are points for which it is negative, then, by Theorem 4 of Ref. 8, the equilibrium point is unstable. We shall be interested in the stability of the equilibrium position corresponding to the trivial solution.

Assuming that in the elastic potential energy the displacements u,v,w are independent of one another, it is shown in Ref. 8 that

$$V_{EL} = \tfrac{1}{2} \int_{D_e} (u\mathscr{L}_u[u] + v\mathscr{L}_v[v] + w\mathscr{L}_w[w]) dD_e$$
$$\geq \tfrac{1}{2} \int_{D_e} \rho\{u\}^T[\Lambda_1{}^2]\{u\} dD_e \tag{9}$$

where ρ is the mass density, $\{u\}$ the column matrix of the elastic displacements u,v,w, and $[\Lambda_1{}^2]$ is the diagonal matrix of the lowest eigenvalues associated with these displacements.

Hence, let us introduce the functional

$$\kappa = T + \tfrac{1}{2} \int_{D_e} \rho\{u\}^T[\Lambda_1{}^2]\{u\} dD_e \tag{10}$$

Now, because $H \geq \kappa$, from our Stability Theorem the equilibrium solution is asymptotically stable if κ is positive definite.

Torque-Free Systems

Let us assume that the system considered is free of external forces, so that the three torque components about the mass center c are zero. It follows that the angular momentum vector about c is conserved, and hence, it represents a motion integral.

From Ref. 9, we conclude that, taking the momentum integral into account, the functional κ can be written in the form $\kappa = \kappa_1 + \kappa_2$, in which $\kappa_1 = T_2$ and

$$\kappa_2 = T_0 + \tfrac{1}{2} \int_{D_e} \rho\{u\}^T[\Lambda_1{}^2]\{u\} dD_e$$
$$= \tfrac{1}{2}\beta^2\{l\}^T[J]^{-1}\{l\} + \tfrac{1}{2} \int_{D_e} \rho\{u\}^T[\Lambda_1{}^2]\{u\} dD_e \tag{11}$$

where $\{l\}$ is the column matrix of the direction cosines $l_{\xi Z}$, $l_{\eta Z}$, $l_{\zeta Z}$ between Z and axes ξ,η,ζ, respectively. The functional κ_2 can be regarded as a *modified dynamic potential*. By virtue of inequality (9), we conclude that κ_2 is in general smaller than (or equal to) the ordinary dynamic potential $T_0 + V_{EL}$.

Since κ_1 is a quadratic functional in the generalized velocities, and κ_2 depends only on the generalized coordinates, κ is positive definite if and only if κ_1 and κ_2 are both positive definite. By definition the quadratic part of the kinetic energy T_2 is positive definite, hence we conclude that if κ_2 is positive definite κ is positive definite.

It is not difficult to show that the matrix $[J]$ can be written as the sum of two matrices $[J]_0$ and $[J]_1$, where $[J]_0$ denotes the inertia matrix about axes x,y,z of the body in undeformed state and $[J]_1$ represents the change in the inertia matrix due to the elastic displacements about ξ,η,ζ as well as the change in the inertia matrix of the undeformed body due to the translations x_c,y_c,z_c, of the origin. In view of this it is shown in Ref. 9 that

$$[K] = [J]^{-1} \cong [J]_0{}^{-1} - [J]_0{}^{-1}[J]_1[J]_0{}^{-1} + [J]_0{}^{-1}[J]_1[J]_0{}^{-1}[J]_1[J]_0{}^{-1}$$

where $[K]$ denotes the inverse of $[J]$. We may therefore express our testing functional in the form

$$\kappa_2 = \tfrac{1}{2}\beta^2\{l\}^T[K]\{l\} + \tfrac{1}{2} \int_{D_e} \{u\}^T[\Lambda_1{}^2]\{u\} dD_e \tag{12}$$

where $[K]$ has the form given above.

The problem of investigating stability reduces to that of testing expression (12) for sign definiteness. To this end, we expand κ_2 in the neighborhood of an equilibrium point E and ignore terms of order greater than two. This process leaves us with a quadratic expression in the generalized coordinates, where the expression is denoted by $\kappa_2|_E$.

However, the generalized coordinates representing the elastic displacements appear in integrals defined over the elastic domain, which precludes its testing for sign definiteness by standard means. This problem can be circumvented through the use of modal analysis in conjunction with series truncation. To this end, we must solve the eigenvalue problems associated with the elastic displacements u,v,w, and represent these displacements by finite series of corresponding eigenfunctions multiplying associated generalized coordinates, where the first depend on spatial coordinates alone and the latter on time alone. Now we are in the position to perform integrations with respect to the spatial variables and write $\kappa_2|_E$ as a quadratic form in the newly defined generalized coordinates. We can define the Hessian matrix $[\mathscr{H}]_E$ corresponding to this quadratic expression, and it should be noted that the order of the Hessian matrix depends on the number of eigenfunctions used in the series representing the elastic displacements. The sign definiteness of $[\mathscr{H}]_E$ may be ascertained by means of Sylvester's criterion (Ref. 1, Sec 6.7). An alternative approach to testing the sign definiteness of $\kappa_2|_E$ involves defining new coordinates representing certain integrals appearing in $\kappa_2|_E$ and using Schwarz's inequality for functions to discretize $\kappa_2|_E$. In general this procedure involves considerably less effort than using modal analysis and yields sharper stability criteria.

In the following section we formulate the stability problem of a torque-free satellite with flexible parts. Subsequently the two techniques described above are used to test the system stability.

Stability of High-Spin Motion of a Satellite with Flexible Appendages

The general theory developed in the preceding sections will now be used to investigate the stability of a satellite simulated by a main rigid body and six flexible thin rods, as shown in Fig. 1. In the undeformed state the body possesses principal moments of inertia A,B,C about axes x,y,z, respectively, and the rods are aligned with these axes. The body is initially spinning undeformed about axis z with angular velocity Ω_s. The domain of the elastic continuum D_e consists of three subdomains D_x,D_y,D_z, bounded by S_x,S_y,S_z, where

$$D_x: -(h_x+l_x)<x<-h_x, h_x<x<(h_x+l_x), \qquad S_x=\pm h_x, \pm(h_x+l_x)$$

$$D_y: -(h_y+l_y)<y<-h_y, h_y<y<(h_y+l_y), \qquad S_y=\pm h_y, \pm(h_y+l_y)$$

$$D_z: -(h_z+l_z)<z<-h_z, h_z<z<(h_z+l_z), \qquad S_z=\pm h_z, \pm(h_z+l_z)$$

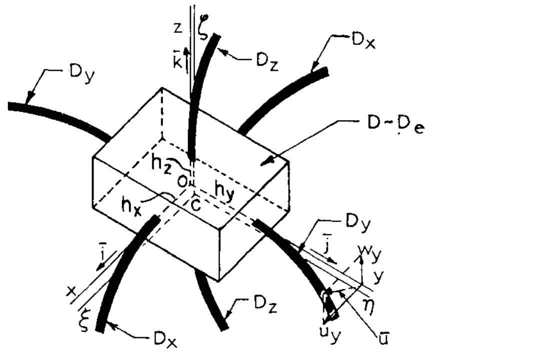

Fig. 1 The rigid satellite with flexible appendages.

Hence $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ over $D-D_e$, $\mathbf{r}=x\mathbf{i}$ over D_x, $\mathbf{r}=y\mathbf{j}$ over D_y, and $\mathbf{r}=z\mathbf{k}$ over D_z. Assuming only flexural transverse vibrations, it follows that

$$\mathbf{u}=\mathbf{u}_x=v_x\mathbf{j}+w_x\mathbf{k},\ \mathbf{u}_c=\mathbf{u}_{cx}=v_{cx}\mathbf{j}+w_{cx}\mathbf{k}, \qquad \mathbf{r}_c=y_c\mathbf{j}+z_c\mathbf{k} \text{ over } D_x$$

$$\mathbf{u}=\mathbf{u}_y=u_y\mathbf{i}+w_y\mathbf{k},\ \mathbf{u}_c=\mathbf{u}_{cy}=u_{cy}\mathbf{i}+w_{cy}\mathbf{k}, \qquad \mathbf{r}_c=x_c\mathbf{i}+z_c\mathbf{k} \text{ over } D_y$$

$$\mathbf{u}=\mathbf{u}_z=u_z\mathbf{i}+v_z\mathbf{j},\ \mathbf{u}_c=\mathbf{u}_{cz}=u_{cz}\mathbf{i}+v_{cz}\mathbf{j}, \qquad \mathbf{r}_c=x_c\mathbf{i}+y_c\mathbf{j} \text{ over } D_z$$

We shall assume that the mass of the rods is symmetrically distributed such that $\rho_x(-x)=\rho_x(x)$, $\rho_y(-y)=\rho_y(y)$, and $\rho_z(-z)=\rho_z(z)$, where ρ_x,ρ_y and ρ_z represent the mass density of the rods associated with the x,y, and z axes, respectively.

The interest lies in investigating the stability of the high-spin motion in which the undeformed satellite rotates with constant angular velocity Ω_s about axis z. Hence, we consider the stability in the neighborhood of the equilibrium point $\theta_1=\theta_2=u_y=u_z=v_x=v_z=w_x=w_y=0$, which in turn, implies $u_{cy}=u_{cz}=v_{cx}=v_{cz}=w_{cx}=w_{cy}=0$. Note that θ_1 and θ_2 represent rotations about ξ and η, respectively.

Since in the equilibrium configuration the body spins about axis z with angular velocity Ω_s, where z coincides with the inertial axis Z, it follows that $\beta=C\Omega_s$. Moreover, it is shown in Ref. 9 that the direction cosines have the values $l_{\xi Z}=-\cos\theta_1\sin\theta_2$, $l_{\eta Z}=\sin\theta_1$, and $l_{\zeta Z}=\cos\theta_1\cos\theta_2$. Introducing all these values into the first term of Eq. (11) and ignoring terms in the generalized coordinates of order larger than two (as well as constant terms), we can write (see Ref. 11)

$$\begin{aligned}\beta^2\{l\}^T[K]\{l\}|_E=\Omega_s^2\bigg[&\frac{C}{B}(C-B)\theta_1^2+\frac{C}{A}(C-A)\theta_2^2-2\frac{C}{A}\times\\ &\theta_2\left(\int_{D_x}\rho_x x w_x dx+\int_{D_z}\rho_z z u_z dz\right)+2\frac{C}{B}\theta_1\left(\int_{D_y}\rho_y y w_y dy+\right.\\ &\left.\int_{D_z}\rho_z z v_z dz\right)-\int_{D_x}\rho_x v_x^2 dx-\int_{D_y}\rho_y u_y^2 dy-\int_{D_z}\rho_z(u_z^2+v_z^2)dz+\\ &\frac{1}{A}\left(\int_{D_x}\rho_x x w_x dx+\int_{D_z}\rho_z z u_z dz\right)^2+\frac{1}{B}\left(\int_{D_y}\rho_y y w_y dy+\right.\\ &\left.\int_{D_z}\rho_z z v_z dz\right)^2+(m-m_x-m_z)y_c^2+(m-m_y-m_z)x_c^2\bigg]\end{aligned} \quad (13)$$

where

$$m_x=\int_{D_x}\rho_x dx,\ m_y=\int_{D_y}\rho_y dy,\ m_z=\int_{D_z}\rho_z dz$$

From Eq. (13) we note that the terms involving x_c^2 and y_c^2 are always positive so that, defining a new testing functional $\kappa_3|_E$, which is obtained from $\kappa_2|_E$ by setting $x_c=y_c=0$, we conclude that $\kappa_3|_E\leq\kappa_2|_E$. It is clear that the case where the center of mass motion in the x and y direction is zero, is the most restrictive case and the satisfaction of stability criteria obtained by ignoring this motion ensures stability for cases with arbitrary motion of the mass center. In view of this, in the sequal we shall ignore the motion of the mass center.

a. Modal analysis

In this section we shall consider a testing functional slightly different from $\kappa_3|_E$. Recalling (9), we note that V_{EL} was replaced by a lower bound using Rayleigh's quotient. In using modal analysis this yields no particular advantage and, hence, we consider the testing functional $\kappa_4|_E$ defined by

$$\kappa_4|_E=\tfrac{1}{2}\beta^2\{l\}^T[K]\{l\}|_E+V_{EL} \quad (14)$$

which represents the ordinary dynamic potential evaluated at equilibrium. We note again that in the first term of Eq. (14) the motion of the mass center is ignored. In analogy with previous reasoning, if $\kappa_4|_E$ is positive definite the equilibrium point is asymptotically stable.

We shall now consider the form of the elastic potential energy. To this end, we must take into account the effect of the centrifugal forces (see Ref. 12, Sec 10.4). Because the satellite has significant spin about axis z, whereas the angular velocities about axes x and y are relatively small, the centrifugal forces acting over the domains D_x, D_y, and D_z are all different. First we wish to distinguish between in-plane and out-of-plane vibrations of the rods associated with domains D_x and D_y. Moreover, we must distinguish between axial and transverse components of the centrifugal forces. It is not difficult to show that domains D_x and D_y are subjected to the axial component of the centrifugal force alone for the out-of-plane vibration and to both the axial and transverse components for the in-plane vibration. On the other hand, domain D_z is subjected to the transverse component alone. The transverse components are accounted for in that part of the kinetic energy not involving velocities, so that only the axial centrifugal forces must be included in the elastic potential energy. Hence, the potential energy can be written in the form

$$V_{EL} = \tfrac{1}{2}\Bigg\{\int_{D_x}\left[v_x \frac{\partial^2}{\partial x^2}\left(EI_{v_x}\frac{\partial^2 v_x}{\partial x^2}\right) + w_x \frac{\partial^2}{\partial x^2}\left(EI_{w_x}\frac{\partial^2 w_x}{\partial x^2}\right)\right]dx - \int_{D_x}\left[v_x\frac{\partial}{\partial x}\left(P_x\frac{\partial v_x}{\partial x}\right) + w_x\frac{\partial}{\partial x}\left(P_x\frac{\partial w_x}{\partial x}\right)\right]dx + \int_{D_y}\left[u_y \frac{\partial^2}{\partial y^2}\left(EI_{u_y}\frac{\partial^2 u_y}{\partial y^2}\right) + w_y \frac{\partial^2}{\partial y^2}\left(EI_{w_y}\frac{\partial^2 w_y}{\partial y^2}\right)\right]dy - \int_{D_y}\left[u_y\frac{\partial}{\partial y}\left(P_y\frac{\partial u_y}{\partial y}\right) + w_y\frac{\partial}{dy}\left(P_y\frac{\partial w_y}{\partial y}\right)\right]dy + \int_{D_z}\left[u_z \frac{\partial^2}{\partial z^2}\left(EI_{u_z}\frac{\partial^2 u_z}{\partial z^2}\right) + v_z \frac{\partial^2}{\partial z^2}\left(EI_{v_z}\frac{\partial^2 v_z}{\partial z^2}\right)\right]dz\Bigg\} \tag{15}$$

where the boundary conditions have been considered (see Ref. 11). The complete expression of $\kappa_4|_E$ is obtained by inserting expression (15) into (14).

To perform a stability analysis by modal approach, we represent the elastic displacements by the following series

$$\left.\begin{aligned} v_x &= \sum_{i=1}^{o_x}\phi_{xoi}(x)V_{xoi}(t) + \sum_{i=1}^{e_x}\phi_{xei}(x)V_{xei}(t) \\ w_x &= \sum_{i=1}^{o_x}\psi_{xoi}(x)W_{xoi}(t) + \sum_{i=1}^{e_x}\psi_{xei}(x)W_{xei}(t)\end{aligned}\right\}\text{ over } D_x \tag{16a}$$

$$\left.\begin{aligned} u_y &= \sum_{i=1}^{o_y}\phi_{yoi}(y)U_{yoi}(t) + \sum_{i=1}^{e_y}\phi_{yei}(y)U_{yei}(t) \\ w_y &= \sum_{i=1}^{o_y}\psi_{yoi}(y)W_{yoi}(t) + \sum_{i=1}^{e_y}\psi_{yei}(y)W_{yei}(t)\end{aligned}\right\}\text{ over } D_y \tag{16b}$$

$$\left.\begin{aligned} u_z &= \sum_{i=1}^{o_z}\phi_{zoi}(z)U_{zoi}(t) + \sum_{i=1}^{e_z}\phi_{zei}(z)U_{zei}(t) \\ v_z &= \sum_{i=1}^{o_z}\psi_{zoi}(z)V_{zoi}(t) + \sum_{i=1}^{e_z}\psi_{zei}(z)V_{zei}(t)\end{aligned}\right\}\text{ over } D_z \tag{16c}$$

where o_x, e_x, o_y, e_y, o_z, e_z are constant integers, ϕ_{xoi}, ϕ_{xei}, $\psi_{xoi}, \ldots, \psi_{zei}$ are eigenfunctions associated with the elastic rods, and V_{xoi}, V_{xei}, $W_{xoi}, \ldots, V_{zei}$ are corresponding generalized coordinates, in which the letters o and e designate odd and even modes of deformation, respectively. The functions ϕ_{xoi}, ϕ_{xei}, $\psi_{xoi}, \ldots, \psi_{xei}$ satisfy certain eigenvalue problems, the details of which can be found in Ref. 11. In view of the above, a typical term in expression (15) becomes

$$\int_{D_x} w_x\left[\frac{\partial^2}{\partial x^2}\left(EI_{w_x}\frac{\partial^2 w_x}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(P_x\frac{\partial w_x}{\partial x}\right)\right]dx = 2\left(\sum_{i=1}^{o_x}\Lambda_{wxi}^2 W_{xoi}^2 + \sum_{i=1}^{e_x}\Lambda_{wxi}^2 W_{xei}^2\right) \tag{17}$$

where Λ_{wxi} are natural frequencies associated with the vibration w_x. [Note that, for consistency with expressions (16), the subscript wx and the index i in Λ_{wxi} are in opposite order from that used in expressions (9) and (10)]. Hence, the potential energy V_{EL} can be regarded as a function of the generalized coordinates V_{xoi}, V_{xei}, W_{xoi}, etc. It follows that $\kappa_4|_E$, Eq. (14), is a quadratic form in the $2(1 + o_x + e_x + o_y + \ldots + e_z)$ variables θ_1, θ_2, U_{yoi}, U_{yei}, $U_{zoi}, \ldots,$. For stability, $\kappa_4|_E$ must be positive definite in these variables. Furthermore, by using even and odd modes to represent the elastic displacements no coupling between even and odd modes occurs. Hence, $\kappa_4|_E$ may be represented as the sum of two independent quadratic forms, $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$, where $\kappa_{4e}|_E$ involves only even modes and $\kappa_{4o}|_E$ involves odd modes and the rigid body motion. Therefore, we have $\kappa_4|_E = \kappa_{4o}|_E + \kappa_{4e}|_E$, where

$$\kappa_{4e}|_E = \sum_{i=1}^{e_x}[(\Lambda_{vxi}^2 - \Omega_s^2)V_{xei}^2 + \Lambda_{wxi}^2 W_{xei}^2] + \sum_{i=1}^{e_y}[(\Lambda_{uyi}^2 - \Omega_s^2)U_{yei}^2 + \Lambda_{wyi}^2 W_{yei}^2] + \sum_{i=1}^{e_z}[(\Lambda_{vzi}^2 - \Omega_s^2)V_{zei}^2 + (\Lambda_{uzi}^2 - \Omega_s^2)U_{zei}^2] \tag{18a}$$

and

$$\begin{aligned}\kappa_{4o}|_E = \tfrac{1}{2}\Omega_s^2\Bigg[&\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 + 4\theta_1\frac{C}{B}\left(\sum_{i=1}^{o_y}J_{wyi}W_{yoi} + \sum_{i=1}^{o_z}J_{vzi}V_{zoi}\right) - 4\theta_2\frac{C}{A}\left(\sum_{i=1}^{o_x}J_{wxi}W_{xoi} + \sum_{i=1}^{o_z}J_{uzi}U_{zoi}\right)\Bigg] + \\ &\sum_{i=1}^{o_x}\sum_{j=i}^{o_x}\left(\Lambda_{wxi}^2\delta_{ij} + \frac{2}{A}\Omega_s^2 J_{wxi}J_{wxj}\right)W_{xoi}W_{xoj} + \sum_{i=1}^{o_y}\sum_{j=1}^{o_y}\left(\Lambda_{wyi}^2\delta_{ij} + \frac{2}{B}\Omega_s^2 J_{wyi}J_{wyj}\right)W_{yoi}W_{yoj} + \\ &\sum_{i=1}^{o_z}\sum_{j=1}^{o_z}\left[(\Lambda_{uzi}^2 - \Omega_s^2)\delta_{ij} + \frac{2}{A}\Omega_s^2 J_{uzi}J_{uzj}\right]U_{zoi}U_{zoj} + \sum_{i=1}^{o_z}\sum_{j=1}^{o_z}\left[(\Lambda_{vzi}^2 - \Omega_s^2)\delta_{ij} + \frac{2}{B}\Omega_s^2 J_{vzi}J_{vzj}\right]V_{zoi}V_{zoj} + \\ &\frac{4}{A}\Omega_s^2\sum_{i=1}^{o_x}\sum_{j=1}^{o_z}J_{wxi}J_{uzj}W_{xoi}U_{zoj} + \sum_{i=1}^{o_x}(\Lambda_{vxi}^2 - \Omega_s^2)V_{xoi}^2 + \\ &\frac{4}{B}\Omega_s^2\sum_{i=1}^{o_y}\sum_{j=1}^{o_z}J_{wyi}J_{vzj}W_{yoi}V_{zoj} + \sum_{i=1}^{o_v}(\Lambda_{uyi}^2 - \Omega_s^2)U_{yoi}^2\end{aligned} \tag{18b}$$

in which

$$J_{vxi} = \int_{h_x}^{h_x + l_x}\rho_x x\phi_{xoi}dx$$

similar expressions hold for J_{wxi}, $J_{uyi}, \ldots, J_{vzi}$. We recall that $\kappa_4|_E$ must be positive definite for the equilibrium point to be asymptotically stable. But $\kappa_4|_E$ can be written as the sum of two independent parts, $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$. It follows that for $\kappa_4|_E$ to be positive definite it is necessary that both $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$ be positive definite.

The expressions for $\kappa_{4e}|_E$ and $\kappa_{4o}|_E$ can be written in the general form

$$\kappa_{4o}|_E = \tfrac{1}{2}\sum_{i=1}^{n_o}\sum_{j=1}^{n_o}\alpha_{oij}q_{oi}q_{oj}$$

$$\kappa_{4e}|_E = \tfrac{1}{2}\sum_{i=1}^{n_e}\sum_{j=1}^{n_e}\alpha_{eij}q_{ei}q_{ej} \tag{19}$$

where q_{oi} and q_{ei} are generalized coordinates and n_o and n_e are integers denoting the number of coordinates q_{ei} and q_{oi} considered. The integers n_o and n_e depend on the number of modes assumed, and hence, on the integers $e_x, e_y, e_z, o_x, o_y, o_z$. The quantities α_{eij} and α_{oij} represent constant coefficients. According to Sylvester's criterion $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$ are positive definite if the conditions

$$|\alpha_{oij}| > 0, \quad |\alpha_{eij}| > 0, i,j = 1, 2, \ldots, k; \quad k = 1, 2, \ldots, n \tag{20}$$

are satisfied, where $|\alpha_{oij}|$ and $|\alpha_{eij}|$ are the principal minor determinants associated with matrices $[\alpha_o]$ and $[\alpha_e]$ of the coefficients. Matrices $[\alpha_o]$ and $[\alpha_e]$ are referred to as Hessian matrices.

Using Eq. (18a) we may write the Hessian matrix $[\alpha_e]$ in the diagonal form $[\alpha_e] = [a_i]$, where the order of the matrix is $n_e = 2e_x + 2e_y + 2e_z$. For a diagonal matrix, conditions (20) merely imply that each diagonal element must be positive. It is shown in Ref. 11 that $\kappa_{4e}|_E$ is positive definite if

$$\Lambda_{vx1} > \Omega_s, \quad \Lambda_{uy1} > \Omega_s \tag{21a}$$

$$\Lambda_{vz1} > \Omega_s, \quad \Lambda_{uz1} > \Omega_s \tag{21b}$$

It is possible to show that conditions (21a) are always met (Ref. 11, Appendix A), so that the testing function $\kappa_{4e}|_E$ is positive definite if conditions (21b) are satisfied.

In terms of the system parameters, it is shown in Ref. 11 that $\kappa_{4o}|_E$ is positive definite if

$$C > A \tag{22a}$$

$$\left(\frac{\Lambda_{wx1}}{\Omega_s}\right)^2 > \frac{2J_{wx1}^2}{C - A - 2\sum_{i=2}^{ox} J_{wxi}^2/(\Lambda_{wxi}/\Omega_s)^2} \tag{22b}$$

$$\left(\frac{\Lambda_{uz1}}{\Omega_s}\right)^2 > 1 + \frac{2J_{uz1}^2}{C - A - 2\sum_{i=2}^{oz} J_{uzi}^2/(\Lambda_{uzi}^2/\Omega_s^2 - 1) - 2\sum_{i=1}^{ox} J_{wxi}^2/(\Lambda_{wxi}/\Omega_s)^2} \tag{22c}$$

and

$$C > B \tag{23a}$$

$$\left(\frac{\Lambda_{wy1}}{\Omega_s}\right)^2 > \frac{2J_{wy1}^2}{C - B - 2\sum_{i=2}^{oy} J_{wyi}^2/(\Lambda_{wyi}/\Omega_s)^2} \tag{23b}$$

$$\left(\frac{\Lambda_{vz1}}{\Omega_s}\right)^2 > 1 + \frac{2J_{vz1}^2}{C - B - 2\sum_{i=2}^{oz} J_{vzi}^2/(\Lambda_{vzi}^2/\Omega_s^2 - 1) - 2\sum_{i=1}^{oz} J_{wyi}^2/(\Lambda_{wyi}/\Omega_s)^2} \tag{23c}$$

In view of the above, we conclude that $\kappa_4|_E$ is positive definite if conditions (21b), (22) and (23) are satisfied. However, we notice that conditions (21b) are contained in (22c) and (23c), so that conditions (22) and (23) present a complete stability picture. It should be noted that, although in general conditions (22) and (23) apply to a satellite with three pairs of symmetric flexible rods, they can also be applied to a satellite containing any smaller number of symmetric pairs of these rods. To this end, we note that if the length of any pair of rods becomes zero the corresponding $J_{vxi}, J_{wxi}, \ldots$, or J_{vzi} becomes zero and the series representing that particular pair of rods is identically zero. Finally, it should be noted that inequalities (22) and (23) have identical forms, so that in the stability criteria established considering inequalities (22) we can substitute for the parameters A, Λ_{uzi}, Λ_{wxi}, J_{uzi} and J_{wxi} the parameters B, Λ_{vzi}, Λ_{wyi}, J_{vzi} and J_{wyi} to obtain stability criteria corresponding to inequalities (23). In the sequel we shall be concerned only with inequalities (22) and results obtained using these inequalities will be applied to inequalities (23) using the substitutions defined above.

A check as to whether inequalities (22) are satisfied will be performed numerically. For convenience we write inequalities (22b) and (22c) in the slightly different form

$$\frac{\Omega_s}{\Lambda_{wx1}} < \left[\frac{(C-A)/A_0 - 2\sum_{i=2}^{ox}(J_{wxi}^2/A_0)/(\Lambda_{wxi}/\Omega_s)^2}{2J_{wx1}^2/A_0}\right]^{1/2} \tag{24a}$$

$$\frac{\Omega_s}{\Lambda_{uz1}} < \left[1 + \frac{2J_{uz1}^2/A_0}{(C-A)/A_0 - R}\right]^{-1/2} \tag{24b}$$

in which $(C-A)/A_0 = (C_0/A_0) - 1 + R_{AX} - R_{AZ}$ and the parameter R is given by

$$R = 2\sum_{i=2}^{oz}(J_{uzi}^2/A_0)/(\Lambda_{uzi}^2/\Omega_s^2 - 1) + 2\sum_{i=1}^{ox}(J_{wxi}^2/A_0)/(\Lambda_{wxi}/\Omega_s)^2 \tag{25}$$

where C_0 and A_0 represent moments of inertia of the rigid part of the satellite about the axes z and x, respectively. The quantities R_{AX} and R_{AZ} represent the ratios

$$R_{AX} = (1/A_0)\int_{D_x}\rho_x x^2 dx, \qquad R_{AZ} = (1/A_0)\int_{D_z}\rho_z z^2 dz$$

At this point, a few comments about the nature of the stability criteria resulting from inequalities (22a) and (24) are in order. We note from (22a) that for stability the spin should be imparted about the axis of maximum moment of inertia. Inequalities (24) indicate that the frequency ratios Ω_s/Λ_{wx1} and Ω_s/Λ_{uz1} are determined by the system parameters and in particular, that Ω_s/Λ_{uz1} must not merely be less than unity as predicted by (21b) but its value must be according to (24b).

b. Method of Integral Coordinates

The stability analysis of the previous section has the disadvantage of leading to an involved numerical procedure. The effects of changes in various system parameters are not easily assessed. Moreover, in using the normal mode approach, the question as to the effect of series truncation on the accuracy of the results remains unanswered. For these reasons we shall seek closed-form stability criteria. Because $\kappa_3|_E \leq \kappa_2|_E$, for asymptotic stability the function $\kappa_3|_E$ must be positive definite, where $\kappa_3|_E$ is given by

$$\begin{aligned}\kappa_3|_E = \tfrac{1}{2}\Big\{&\Omega_s^2\Big[\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 - 2\frac{C}{A}\theta_2 \times \\ &\Big(\int_{D_x}\rho_x x w_x dx + \int_{D_z}\rho_z z u_z dz\Big) + 2\frac{C}{B}\theta_1 \times \\ &\Big(\int_{D_y}\rho_y y w_y dy + \int_{D_z}\rho_z z v_z dz\Big) + \frac{1}{A}\Big(\int_{D_x}\rho_x x w_x dx + \\ &\int_{D_z}\rho_z z u_z dz\Big)^2 + \frac{1}{B}\Big(\int_{D_y}\rho_y y w_y dy + \int_{D_z}\rho_z z v_z dz\Big)^2\Big] + \\ &\int_{D_x}\rho_x\Lambda_{wx1}^2 w_x^2 dx + \int_{D_x}\rho_x(\Lambda_{vx1}^2 - \Omega_s^2)v_x^2 dx + \\ &\int_{D_y}\rho_y\Lambda_{wy1}^2 w_y^2 dy + \int_{D_y}\rho_y(\Lambda_{uy1}^2 - \Omega_s^2)u_y^2 dy + \\ &\int_{D_z}\rho_z(\Lambda_{uz1}^2 - \Omega_s^2)u_z^2 dz + \int_{D_z}\rho_z(\Lambda_{vz1}^2 - \Omega_s^2)v_z^2 dz\Big\}\end{aligned} \tag{26}$$

Again we note that $\kappa_3|_E$ is both a function and a functional and it may not be possible to determine its sign definiteness by standard techniques. However, by defining suitable new coordinates and using Schwarz's inequality for functions, it may be possible to circumvent this problem. To this end, we define the following integral coordinate‡

$$\bar{v}_x(t) = \int_{D_x} \rho_x x v_x(x,t)dx \quad (27)$$

with analogously defined integral coordinates $\bar{w}_x(t)$, $\bar{u}_y(t), \ldots, \bar{v}_z(t)$. Using Schwarz's inequality, we have

$$\left(\int_{D_x} \rho_x x v_x dx\right)^2 \leq \int_{D_x} \rho_x x^2 dx \int_{D_x} \rho_x v_x^2 dx \quad (28)$$

Recalling the definition of $\bar{v}_x$, and solving for

$$\int_{D_x} \rho_x v_x^2 dx$$

inequality (28) yields

$$\int_{D_x} \rho_x v_x^2(x,t)dx \geq \bar{v}_x^2(t)/A_0 R_{AX} \quad (29)$$

Similar inequalities can be written for the integrals

$$\int_{D_x} \rho_x w_x^2(x,t)dx,\ \int_{D_y} \rho_y w_y^2(y,t)dy, \ldots, \int_{D_z} \rho_z v_z^2(z,t)dz$$

where the integrals over D_y and D_z involve the ratios

$$R_{BY} = (1/B_0)\int_{D_y} \rho_y y^2 dy,\ R_{BZ} = (1/B_0)\int_{D_z} \rho_z z^2 dz,$$

in which B_0 denotes the mass moment of inertia of the rigid part of the satellite about the y axis. Inserting the appropriate inequalities into Eq. (26), noting that $\Lambda_{vx1} > \Omega_s$ and $\Lambda_{uy1} > \Omega_s$, and if, in addition, we assume $\Lambda_{uz1} > \Omega_s$ and $\Lambda_{vz1} > \Omega_s$ (which will later be shown to be the case), then we can define a new testing function $\kappa_5|_E$ given by

$$\kappa_5|_E = \tfrac{1}{2}\Bigg\{\Omega_s^2\Bigg[\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 - 2\frac{C}{A}\theta_2(\bar{w}_x + \bar{u}_z) + 2\frac{C}{B}\theta_1(\bar{w}_y + \bar{v}_z) + \frac{1}{A}(\bar{w}_x + \bar{u}_z)^2 + \frac{1}{B}(\bar{w}_y + \bar{v}_z)^2\Bigg] + \frac{\Lambda_{wx1}^2}{A_0 R_{AX}}\bar{w}_x^2 + \frac{\Lambda_{wx1}^2}{B_0 R_{BY}}\bar{w}_y^2 + \frac{(\Lambda_{vx1}^2 - \Omega_s^2)}{A_0 R_{AX}}\bar{v}_x^2 + \frac{(\Lambda_{uy1}^2 - \Omega_s^2)}{B_0 R_{BY}}\bar{u}_y^2 + \frac{(\Lambda_{uz1}^2 - \Omega_s^2)}{A_0 R_{AZ}}\bar{u}_z^2 + \frac{(\Lambda_{vz1}^2 - \Omega_s^2)}{B_0 R_{BZ}}\bar{v}_z^2\Bigg\} \quad (30)$$

where $\kappa_5|_E \leq \kappa_3|_E$. Hence, if $\kappa_5|_E$ is positive definite the equilibrium point is asymptotically stable. We note that $\kappa_5|_E$ can be written as the sum of three independent quadratic forms, each of which must be positive definite. Denoting these forms by $\kappa_{51}|_E$, $\kappa_{52}|_E$, $\kappa_{53}|_E$ and their associated Hessian matrices by $[\mathscr{H}_{51}]_E$, $[\mathscr{H}_{52}]_E$, $[\mathscr{H}_{53}]_E$, respectively, we obtain

$$[\mathscr{H}_{51}]_E = \tfrac{1}{2}\begin{bmatrix} \dfrac{(\Lambda_{vx1}^2 - \Omega_s^2)}{A_0 R_{AX}} & 0 \\ 0 & \dfrac{(\Lambda_{uy1}^2 - \Omega_s^2)}{B_0 R_{BY}} \end{bmatrix} \quad (31a)$$

‡ After this paper had been written the authors became aware of a somewhat similar method used in conjunction with the stability of rotating bodies containing fluids (see Ref. 13).

$$[\mathscr{H}_{52}]_E = \frac{\Omega_s^2}{2A}\begin{bmatrix} C(C-A) & -C & -C \\ -C & \dfrac{\Lambda_{wx1}^2 A}{\Omega_s^2 A_0 R_{AX}} + 1 & 1 \\ -C & 1 & \left(\dfrac{\Lambda_{uz1}^2}{\Omega_s^2} - 1\right)\dfrac{A}{A_0 R_{AZ}} + 1 \end{bmatrix} \quad (31b)$$

$$[\mathscr{H}_{53}]_E = \frac{\Omega_s^2}{2B}\begin{bmatrix} C(C-B) & C & C \\ C & \dfrac{\Lambda_{wy1}^2 B}{\Omega_s^2 B_0 R_{BY}} + 1 & 1 \\ C & 1 & \left(\dfrac{\Lambda_{vz1}^2}{\Omega_s^2} - 1\right)\dfrac{B}{B_0 R_{BZ}} + 1 \end{bmatrix} \quad (31c)$$

An application of Sylvester's criterion to matrices (31), yields the following stability criteria

$$\Lambda_{vx1}^2 > \Omega_s^2,\ \Lambda_{uy1}^2 > \Omega_s^2 \quad (32a)$$

$$C > A$$
$$(\Lambda_{wx1}/\Omega_s)^2 > A_0 R_{AX}/(C-A) \quad (32b)$$
$$\left(\frac{\Lambda_{uz1}}{\Omega_s}\right)^2 > 1 + \frac{(\Lambda_{wx1}/\Omega_s)^2 A_0 R_{AZ}}{(C-A)(\Lambda_{wx1}/\Omega_s)^2 - A_0 R_{AX}}$$

and

$$C > B$$
$$(\Lambda_{wy1}/\Omega_s)^2 > B_0 R_{BY}/(C-B) \quad (32c)$$
$$\left(\frac{\Lambda_{vz1}}{\Omega_s}\right)^2 > 1 + \frac{(\Lambda_{wy1}/\Omega_s)^2 B_0 R_{BZ}}{(C-B)(\Lambda_{wy1}/\Omega_s)^2 - B_0 R_{BY}}$$

respectively. From our previous discussion we conclude that inequalities (32a) are always satisfied. Furthermore, we note that inequalities (32b) and (32c) possess identical forms. In view of that, we shall establish stability criteria using inequalities (32b) and replace the parameters A, A_0, R_{AX}, R_{AZ}, Λ_{wx1}/Ω_s and Λ_{uz1}/Ω_s by B, B_0, R_{BY}, R_{BZ}, Λ_{wy1}/Ω_s and Λ_{vz1}/Ω_s, respectively, to derive criteria valid for Eq. (32c). For convenience, inequalities (32b) are written in the slightly different form

$$C > A \quad (33a)$$

$$\frac{\Omega_s}{\Lambda_{wx1}} < \left[\frac{C_0/A_0 + R_{AX} - 1 - R_{AZ}}{R_{AX}}\right]^{1/2} \quad (33b)$$

$$\frac{\Omega_s}{\Lambda_{uz1}} < \left[1 + \frac{R_{AZ}}{C_0/A_0 - R_{AZ} - 1 + R_{AX}(1 - \Omega_s^2/\Lambda_{wx1}^2)}\right]^{-1/2} \quad (33c)$$

Three major conclusions can be drawn from inequalities (33): a) For spin stabilization the spinning motion should be imparted about the axis of maximum moment of inertia. b) Spin stabilization is possible if the spin ratios Ω_s/Λ_{wx1} and Ω_s/Λ_{uz1} satisfy inequalities (33b) and (33c), which involve the system parameters R_{AX}, R_{AZ} and C_0/A_0. In addition the frequency ratio Ω_s/Λ_{uz1} should not exceed unity. c) A satellite which is stable without radial rods remains stable if radial rods are added.

To verify the last statement, we recall that Λ_{wx1} represents the first natural frequency of the out-of-plane vibration of a rotating rod and it must be greater than Ω_s, so that

$$(\Omega_s/\Lambda_{wx1}) < 1$$

In addition, for a satellite with no radial rods, we find from inequality (33a), that for stability we should have $(C_0/A_0) > 1 + R_{Az}$. Using these results, we see that inequality (33b) yields a less stringent criterion, as the right side of (33b) is always greater than unity. Moreover, for any value of R_{AX} other than zero inequality (33c) is less restrictive than the same inequality with $R_{AX} = 0$.

We note that, by contrast with inequalities (24), the evaluation of criteria (33) requires much less numerical work. In particular, for inequalities (24) we must solve a certain eigenvalue problem (see Ref. 11), and obtain n frequencies ${}^n\Lambda_{wxi}$ and eigenvectors $\{a^{(i)}\}$, whereas inequalities (33) require only the first natural frequency Λ_{wx1} of the rotating rod.

Numerical Results

The general solution of the stability problem of a rigid satellite with three (or less) pairs of uniform rods has been programmed for digital computation, and a numerical solution has been obtained on an IBM 360 computer. Results are presented for the criteria developed using both modal analysis and integral coordinates. For the numerical study it is assumed that rods x and z have equal mass and stiffness properties, and, in addition, the rigid body dimensions h_x and h_z are equal (see later statement concerning rods y). The above restrictions are placed only on the numerical solution to facilitate the presentation of data; there are no such restrictions placed on either the problem formulation or computer program. In Figs. 2 and 3, the results obtained using modal analysis are represented by the dashed lines and those obtained using integral coordinates by solid lines. Figure 4 shows the value of the ratio $\Omega_s/(\Lambda_{wx1})_{NR}$ vs Ω_s/Λ_{wx1}, where $(\Lambda_{wx1})_{NR}$ is the first natural frequency of the nonrotating rod obtained by setting $\Omega_s = 0$. The first natural frequency of the rotating rod is denoted by Λ_{wx1}. The quantity $HX = h_x/l_x$ plays the role of a parameter. This figure enables us to make use of the parametric plots of Fig. 2 without having to solve the eigenvalue problem for the rotating rods, where Fig. 2 shows the spin ratio Ω_s/Λ_{wx1} required for stability as a

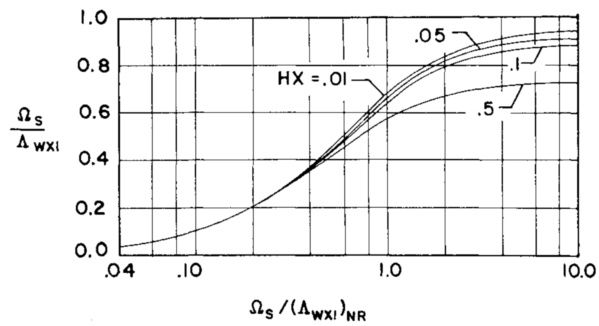

Fig. 4 First natural frequency of rotating vs nonrotating rod with HX as a parameter.

function of $(C_0/A_0) - R_{Az}$, with R_{Ax} as a parameter. The region below the appropriate curve is stable. The curve shows that for $(C_0/A_0) - R_{Az} = 1$ the allowable spin ratio is equal to unity and for $(C_0/A_0) - R_{Az} > 1$ no instability exists. We note in Fig. 4 that the ratio Ω_s/Λ_{wx1} is always less than unity. The extent to which Ω_s/Λ_{wx1} is less than unity depends on the parameter HX, in the sense that if HX increases the ratio Ω_s/Λ_{wx1} decreases. Hence, in Fig. 2 all values of Ω_s/Λ_{wx1} greater than unity are said to be dynamically impossible. However, the dynamically impossible region may include values of Ω_s/Λ_{wx1} considerably less than unity as shown in Fig. 4. It is reiterated that Figs. 2 and 4 are to be used together. Namely, starting with a value of $\Omega_s/(\Lambda_{wx1})_{NR}$, Fig. 4 gives Ω_s/Λ_{wx1}, which is then used in Fig. 2. It should be noted that Figs. 2 and 4 present a complete stability analysis for a satellite with radial rods alone. Figure 3 shows the allowable spin ratio Ω_s/Λ_{uz1} for stability as a function of R_{Az}, with the length ratio l_x/l_z as a parameter. The region below the appropriate curve is stable. These curves show that the allowable spin ratio Ω_s/Λ_{uz1} must be lower than unity; the extent to which it must be lower than unity depends on the system parameters. It should be noted from Fig. 3 that the most restrictive region of stability is associated with the parameter $l_x/l_z = 0$, namely the case in which there are no radial rods. As noted earlier, *any stable satellite possessing axial rods alone will remain stable with the addition of radial rods*. Indeed the addition of radial rods increases the region of stability significantly and for length ratios $l_x/l_z > 10$ the allowable spin ratio is very near unity.

For comparison purposes, a problem which can be considered as a special case of the present one, in the sense that it considers only spin axis rods, has been considered; this is the problem investigated in Ref. 9. Inequality (33c) for the case where R_{AX} equals zero yields the appropriate stability criterion. Results using this criterion as well as results from Ref. 9 are presented in Fig. 5. The results of Ref. 9 working with density

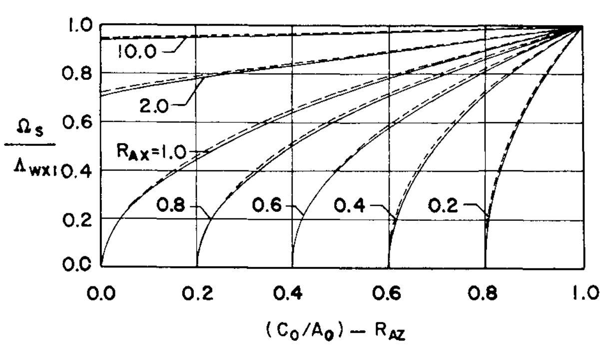

Fig. 2 Stability regions with R_{AX} as a parameter.

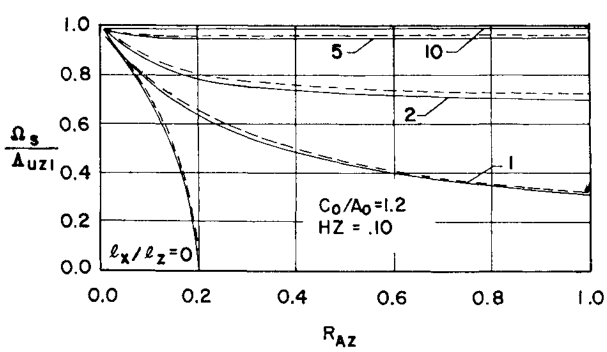

Fig. 3 Stability regions with l_x/l_z as a parameter.

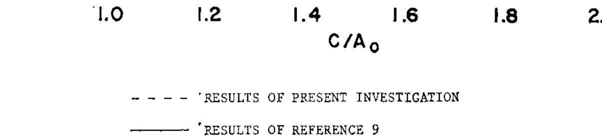

Fig. 5 Comparison of results of present investigation with results of Ref. 9.

functions are more restrictive than those of the present investigation.

It should be noted that diagrams identical in every respect to Figs. 2–5 but with Λ_{vz1}, Λ_{wy1}, B_0, R_{BY}, R_{BZ}, HY and l_y replacing Λ_{uz1}, Λ_{wx1}, A_0, R_{AX}, R_{AZ}, HX and l_x, respectively, may be obtained from inequalities (23) and (32c).

Summary and Conclusions

The mathematical formulation associated with the problem of the stability of motion of a satellite consisting of a main rigid body and three (or less) pairs of flexible rods has been completed. The rods are capable of flexure in two orthogonal directions. Whereas, the rotational motion of the body is described by generalized coordinates depending on time alone, the elastic displacements of the rods depend both on spatial position and time. Because of the elastic motion of the rods, the center of mass of the body is shifting continuously relative to the main rigid body. These displacements, however, do not add degrees of freedom since they can be expressed in terms of integrals involving the elastic displacements. Assuming no external torques, there exist motion integrals in the form of momentum integrals. These integrals can be regarded as constraint equations relating the system velocities.

The Liapunov direct method has been chosen for the stability analysis because it is likely to yield results which can be interpreted more readily than those obtained by a purely numerical integration of the equations of motion. Since the elastic vibrations result in energy dissipation, it is shown that the equilibrium position is asymptotically stable if the Hamiltonian is positive definite and unstable if it can take negative values in the neighborhood of the equilibrium. Determining the sign definiteness of the Hamiltonian is complicated by the fact that no testing density function can be readily defined as in Refs. 8 and 9. Two methods have been presented to deal with this problem. The first, the standard modal analysis in conjunction with series truncation, develops criteria in terms of finite series associated with the natural modes and frequencies of the elastic rods. The second, the method of integral coordinates, yields closed-form stability criteria involving the system parameters such as the body moments of inertia, the length and mass distribution of the rods, the lowest natural frequencies of the rods, and the satellite spin velocity. The advantage of the method of integral coordinates is demonstrated by the relative ease with which closed-form stability criteria are developed and by the amount of information which can be extracted from their ready physical interpretation. In particular, the analysis shows that, for stability, the spinning motion is to be imparted about the axis of maximum moment of inertia and that the allowable spin ratios Ω_s/Λ_{wx1}, Ω_s/Λ_{wy1}, Ω_s/Λ_{uz1} and Ω_s/Λ_{vz1} are determined by the system parameters. The first is recognized as the "greatest moment of inertia" criterion. Moreover, the spin ratios Ω_s/Λ_{uz1} and Ω_s/Λ_{vz1} should not be merely lower than unity (as they should be in the case of simple harmonic excitation of rods to prevent reasonance), but they are further restricted by the system parameters. It is also shown that a stable spinning satellite not containing radial rods will remain stable if radial rods are added.

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